

On the Continuous Dependence of Solutions to Nonlinear Problems in Hilbert Space

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1. INTRODUCTION

Various physical problems lead to the study of equations of the form

$$A\dot{u} = \mathcal{L}(t)u, \quad t \geq 0, \quad (1.1)$$

with

$$u(0) = u_0, \quad (1.2)$$

where A and $\mathcal{L}(t)$ are operators on some suitable specified subspace of a Hilbert space \mathcal{H} and \dot{u} is the derivative with respect to the time t of the vector-valued function $u(t) \in \mathcal{H}$.

It is known that the problem (1.1), (1.2) is not, in general, well posed in the sense of Hadamard [1]. For example, solutions to (1.1), (1.2) might not depend continuously on the initial data u_0 . It is then necessary to know under which conditions continuous dependence can be recovered. An investigation of John [2] has shown that in certain situations it is sufficient to impose a uniform bound on solutions to achieve this. In some of these cases continuity must be replaced by the corresponding concept of Hölder continuity.

In the last years a number of workers [3–10] have given numerous examples of this type of continuous dependence and physical justification for it. The following simple example, considered in [9], may provide an intuitive idea of Hölder stability and the situations in which it might be applicable. Let the solutions of a linear system be such that their time dependence is of the type e^{nt} . Then, the solutions which are bounded at $t = t_1$ would be of the form $e^{-n(t-t_1)}$ and thus remain arbitrarily near the null solution for $t \in [0, t_1)$ but not for $t \in [0, t_1]$. It results that there exists continuous dependence on the initial data in all compact sub-intervals of $[0, t_1)$ but not on the interval $[0, t_1]$ itself.

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In the present paper we are concerned with the Hölder continuous dependence on the initial data of solutions to the problem (1.1), (1.2). More precisely, we show that under certain conditions imposed on A , \mathcal{L} , and u_0 , there exists, for each sufficiently regular solution u of (1.1), (1.2), a time interval $I_0 = [0, T_0)$ such that in every compact sub-interval of I_0 the solution u depends Hölder continuously on u_0 provided it belongs to the class of solutions which admit of a uniform bound (in a sense which will be made precise later). The technique used in proving this result is that of logarithmic convexity (see, e.g., [9, 11, 12]).

2. CONTINUOUS DEPENDENCE

Let \mathcal{H} be a real Hilbert space provided with the norm $\|\cdot\|$ induced by the scalar product $[\cdot, \cdot]$.

Let $\mathcal{D} \subseteq \mathcal{H}$ be a dense domain and let $A: \mathcal{D} \rightarrow \mathcal{H}$ be a linear, symmetric, time-independent operator. Assume that A satisfies

$$[u, Au] > 0, \quad \text{for all } u \in \mathcal{D}, u \neq 0. \quad (2.1)$$

Assume also that the operator $\mathcal{L}(t): \mathcal{D} \rightarrow \mathcal{H}$ satisfies

$$[u, \mathcal{L}(t)u] > 0, \quad \text{for all } u \in \mathcal{D}, t \in (0, T], \quad (2.2)$$

$$[u_0, \mathcal{L}(0)u_0] = 0, \quad (2.3)$$

and $\dot{\mathcal{L}}(t)$ exists in the strong sense for $t \in [0, T]$.

Let $u \in \mathcal{D}$, $u \neq 0$ be a strong solution of the problem (1.1), (1.2) (i.e., a solution for which \dot{u} exists in the strong sense, is continuous, and is \mathcal{D} valued).

Define

$$F(t) \equiv [u, Au], \quad u \in \mathcal{D}, \quad t \in [0, T], \quad (2.4)$$

$$f(t) \equiv \frac{\dot{F}(t)}{F(t)}, \quad t \in [0, T]. \quad (2.5)$$

Using (1.1) and the symmetry of A we see that

$$f(t) = \frac{2[u, A\dot{u}]}{F(t)} = \frac{2[u, \mathcal{L}(t)u]}{F(t)}. \quad (2.6)$$

Our hypotheses imply that f is continuously differentiable with respect to t

in $[0, T]$ and also that $f(t) > 0$, $t \in (0, T]$, $f(0) = 0$. We shall prove that, in this case, there exists an interval $I = [0, T_0]$, $T_0 \leq T$, throughout which

$$\dot{f}(t) = \frac{F\ddot{F} - \dot{F}^2}{F^2} = \frac{d^2}{dt^2} [\ln F(t)] \geq 0. \quad (2.7)$$

If $\dot{f}(0) \neq 0$ the assertion follows by continuity. If $\dot{f}(0) = 0$ we see (by applying the mean value theorem to the function f) that in every interval $(0, \hat{t})$ with $\hat{t} \leq T$ either $f \equiv 0$ or there exists at least one point t^* at which $\dot{f}(t^*) > 0$. The continuity reveals the existence of an interval $(t_1, t_2) \subset (0, \hat{t})$, $t^* \in (t_1, t_2)$ throughout which $\dot{f}(t) > 0$ and either $t_1 = 0$ or $\dot{f}(t_1) = 0$. If $\dot{f}(t) \geq 0$ for $t \in [0, t_1]$ we may take $T_0 = t_2$ and the assertion is proved. Assume then that there exists a point $t^0 \in (0, t_1)$ such that $\dot{f}(t^0) < 0$. It then follows that for every interval $(0, \hat{t})$ there exists a sub-interval (t_1, t_2) , where $\dot{f} > 0$ and also a sub-interval (t_3, t_4) , $t^0 \in (t_3, t_4)$, where $\dot{f} < 0$. Therefore the point $t = 0$ cannot be a point of (strict) extremum for f . Thus, we have reached a contradiction which proves our claim.

By means of expansions in finite Taylor series, or alternatively by Jensen's inequality [13], we obtain from (2.7) the inequality

$$F(t) \leq F(0)^{(T_0-t)/T_0} F(T_0)^{t/T_0}, \quad t \in (0, T_0). \quad (2.8)$$

DEFINITION. If, at time T_0 , the solution u to the problem (1.1), (1.2) is uniformly bounded in the sense that

$$F(T_0) < M, \quad (2.9)$$

for some positive finite constant M , then we say that u belongs to the class \mathcal{M} and denote $u \in \mathcal{M}$.

For $u \in \mathcal{M}$, we deduce from (2.8) that

$$F(t) \leq M^{1-\delta} [F(0)]^\delta, \quad \delta = 1 - t/T_0, \quad 0 \leq \delta < 1. \quad (2.10)$$

The last relation implies the Hölder continuous dependence of the solution u on the initial data u_0 , in the measure $F(t)$, on every compact sub-interval of $I_0 = [0, T_0)$. Thus, we have the following.

THEOREM. Let the operators A , $\mathcal{L}(t)$, and the function u_0 in (1.1), (1.2) satisfy (2.1)–(2.3). Assume that $\mathcal{L}(t)$ exists (in the strong sense) in some time interval $[0, T]$. Then for each strong solution $u \in \mathcal{M}$ there exists a time interval $I_0 = [0, T_0) \subset [0, T]$ such that u depends Hölder continuously on the initial data u_0 , in the measure $F(t)$ on every compact sub-interval of I_0 .

Remark 1. The inequality (2.8) may be written in the form

$$F(t) \leq \exp \left[\frac{t}{T_0} \ln \left(\frac{F(t)}{F(0)} \right) \right], \quad t \in [0, T_0), \quad (2.11)$$

which shows that there exists an interval $I_0 = [0, T_0)$ throughout which $F(t)$ cannot be greater than some time-increasing exponential function.

Remark 2. Suppose that the following supplementary conditions are satisfied:

(a) \mathcal{D} is a connected set included in some ball $B_v = \{u(t) \in \mathcal{H} : \|u\| < v\}$.

(b) $\mathcal{L}(t)$ has a linear Gâteaux differential $(d\mathcal{L}(t)u)h$ at every point of \mathcal{D} and for each $t \geq 0$.

(c) The functional $[(d\mathcal{L}(t)u)h_1, h_2]$ is continuous in u at every point of \mathcal{D} .

Then, every operator $\mathcal{L}(t)$ which satisfies (2.2) can be written in the form [14]

$$\mathcal{L}(t)u = d\Phi(t)u + U(t)u, \quad (2.12)$$

$$\Phi(t)u = \int_0^1 p_t(\lambda u) \frac{d\lambda}{\lambda}, \quad (2.13)$$

where $p_t(u)$ belongs to the class of all functionals on \mathcal{H} that have a continuous first Gâteaux differential satisfying

$$p_t(u) > 0, \quad \text{for } u \neq 0, \quad p_t(0) = 0, \quad (2.14)$$

and where $U(t)$ ranges over all continuous operators such that

$$[U(t)u, u] = 0, \quad u \in \mathcal{H}, \quad (2.15)$$

$$[U(t)0, h] = 0, \quad h \in \mathcal{H}. \quad (2.16)$$

3. EXAMPLES

Here we give two examples demonstrating the applicability of the above theorem.

EXAMPLE 1. Let $\Omega \subset R^n$ be a bounded region with the boundary Γ . Assume that Ω is sufficiently regular and Γ is smooth enough to make

integration by parts and application of the divergence theorem permissible. Let $\mathcal{H} = L^2(\Omega)$ be equipped with the scalar product

$$(u, v) = \int_{\Omega} uv \, d\Omega. \quad (3.1)$$

We consider the initial-boundary value problem

$$\begin{aligned} \dot{u} + t\Delta u &= u^3(e^t - 1), & \text{in } \Omega, \\ u|_{\Gamma} &= 0, & t \geq 0, \\ u(x, 0) &= u_0(x), \end{aligned} \quad (3.2)$$

and choose

$$\mathcal{D} = \{u(t): u(t) \in C^2(\bar{\Omega}), u|_{\Gamma} = 0, t \geq 0\}. \quad (3.3)$$

It is known that \mathcal{D} is a linear set, dense in $L^2(\Omega)$ [15, Chap. IX]. It follows then that in this case all the requirements of the above Theorem are satisfied.

EXAMPLE 2. Let Ω , Γ , and \mathcal{D} be defined as before and let $\mathcal{L}(t)$ be given by

$$\mathcal{L}(t)u = - \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial \Phi(u_{k,l}; t)}{\partial u_{i,j}} \right), \quad i, k, l = 1, 2, \dots, n, \quad (3.4)$$

where $\Phi(\cdot, \cdot)$ is a function which satisfies

$$\Phi(u_{k,l}; \cdot) \in C^1[0, T], \quad \Phi(u_{k,l}; t) \geq 0, \quad t \geq 0, \quad (3.5)$$

$$\frac{\partial \Phi(u_{k,l}; 0)}{\partial u_{i,j}} = 0, \quad (3.6)$$

$$\sum_{i,j=1}^n \left(\frac{\partial \Phi}{\partial u_{i,j}} u_{i,j} \right) \geq \alpha \Phi, \quad \alpha = \text{const}, \alpha > 0, \quad (3.7)$$

and where we have denoted

$$u_{k,l} = \frac{\partial u_k}{\partial x_l}, \quad k, l = 1, 2, \dots, n. \quad (3.8)$$

Consider the following initial-boundary value problem.

$$\begin{aligned} \dot{u}_i + \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial \Phi(u_{k,l}; t)}{\partial u_{i,j}} \right) &= 0, \quad u \in \mathcal{D}, \\ u|_T &= 0, \quad t \geq 0, \\ u(x, 0) &= u_0(x). \end{aligned} \quad (3.9)$$

Integrating by parts and applying the divergence theorem we obtain

$$[u, \mathcal{L}(t)u] = \int_{\Omega} \sum_{i,j=1}^n \left(\frac{\partial \Phi}{\partial u_{i,j}} u_{i,j} \right) d\Omega, \quad u \in \mathcal{D}. \quad (3.10)$$

Now, it is easily seen, from (3.5)–(3.7), that the Theorem in the previous section can be applied.

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